

Quantum toboggans: models exhibiting a multisheeted \mathcal{PT} –symmetry

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Abstract

A generalization of the concept of \mathcal{PT} –symmetric Hamiltonians $H = p^2 + V(x)$ is presented. For the usual analytic potentials $V(x)$ (with singularities) and for the recently widely accepted “ \mathcal{PT} –symmetric” asymptotic boundary conditions for wave functions $\psi(x)$ (selected inside a pair of complex wedges generalizing the usual $x \rightarrow \pm\infty$ asymptotics), non-equivalent quantum toboggans are defined as integrated along topologically different paths \mathcal{C} of coordinates $x \in \mathbb{C}$.

1 Introduction

Among non-Hermitian Hamiltonians $H \neq H^\dagger$ defined, say, in a suitable “auxiliary” Hilbert space $\mathcal{H}^{(aux)}$, a privileged subfamily is formed by their η –pseudohermitian special cases which are such that $H^\dagger = \eta H \eta^{-1}$ in terms of a suitable operator $\eta \neq I$. Whenever $\eta = \mathcal{P}$ happens to coincide with the operator of parity, we arrive at the increasingly popular [1] \mathcal{PT} –symmetric quantum models H .

In general, *all* of the above-mentioned Hamiltonians H can happen to possess a real and discrete spectrum of energies $E_0 \leq E_1 \leq \dots$. In such a case we may always imagine that while our $H \neq H^\dagger$ (acting in $\mathcal{H}^{(aux)}$) is just one of many possible operator representations of such a spectrum of bound-state energies, there may exist a “true” physical Hilbert space $\mathcal{H}^{(phys)}$ and, in it, an orthonormalized basis of kets $\{|n\rangle\}_{n=0,1,\dots}$ such that a certain operator $H^{(phys)}$ defined by its spectral representa-

tion,

$$H^{(phys)} = \sum_{n=0}^{\infty} |n\rangle E_n \langle n| \quad (1)$$

can be declared the “true”, mathematically correct representant of the Hamiltonian of our original quantum system. Indeed, the latter operator is manifestly *self-adjoint* in $\mathcal{H}^{(phys)}$ and all the postulates of Quantum Mechanics are satisfied.

There exist multiple practical applications of such an idea of the use of two different Hilbert spaces in parallel (cf., e.g., the most recent and up-to-date review [1]). One of their most characteristic shared features is that while $H^{(phys)}$ is a technically very complicated operator in $\mathcal{H}^{(phys)}$, its “equivalent” representation H in $\mathcal{H}^{(aux)}$ is, by assumption, “very simple”, in comparison at least [2].

In the latter sentence, we used the quotation marks on purpose: “very simple” need not mean trivial. Even worse, the “equivalence” of the mutual mapping between H and $H^{(phys)}$ is particularly conventional a concept. Indeed, it is obvious that in the very second paragraph of our present text the readers may have noticed that our choice of the basis $\{|n\rangle\}_{n=0,1,\dots}$ was *entirely* arbitrary.

Of course, a responsible attitude towards the latter ambiguity problem unifies various applications of the formalism which may be sampled as ranging from nuclear physics [2] and field theory [1] far beyond the territory of quantum theories, involving even the terrains as distant as random matrices [3], cosmology [4] or classical optics [5], electrodynamics [6] and magnetohydrodynamics [7]. In each of these contexts, the ambiguity problem finds its specific resolution. In particular, in the narrower domain of quantum theories themselves, various additional conditions are usually imposed, relying on various persuasive phenomenological arguments as summarized, in compact form, in refs. [2, 8] (for Quantum Mechanics) and [1] (mainly in the context of field theories).

Once a suitable map between $\mathcal{H}^{(aux)}$ and $\mathcal{H}^{(phys)}$ has been established (this is not to be discussed here), the most persuasive distinction between these spaces can be seen in the prohibitively complicated form of the operator $H^{(phys)}$ when compared with H . Typically, in the specific, \mathcal{PT} –symmetric quantum models $H = \mathcal{P} H^\dagger \mathcal{P}$, the manifestly self-adjoint $H^{(phys)}$ is even fairly difficult to define. In contrast, H is often selected as an ordinary differential operator, easily and efficiently tractable by many standard mathematical techniques. *Pars pro toto*, we recommend the readers to have a look at the “first nontrivial”, exactly solvable \mathcal{PT} –symmetric harmonic oscillator [9] for illustration.

On the latter, \mathcal{PT} –symmetric background there is still a lot of space for the study of some “slightly” more complicated though still mathematically tractable Hamilto-

nians H . In this spirit we intend to pay attention here to the newly introduced [10] and developed [11, 12] family of the models called “quantum toboggans” (QT). Our brief review and introduction to this subject will be divided in section 2 (on the concept of quantum toboggans), section 3 (on QT constructions), section 4 (on the QT models with more branch points) and summary.

2 The concept of quantum toboggans

During a “prehistory” of our theme, several complex potentials $V(x)$ have been shown to generate real bound-state spectra, be it an imaginary cubic $V(x) \sim ix^3$ at $|x| \gg 1$ [13] or a negative quartic $V(x) \sim -x^4$ at $|x| \gg 1$ [14], both exhibiting \mathcal{PT} -symmetry (note that \mathcal{P} changes parity in this context, $x \rightarrow -x$, while the complex conjugation \mathcal{T} mimics time reversal).

In some sense, the “history” commenced in 1993 when Bender and Turbiner published a letter where certain standard ordinary differential Schrödinger equations

$$\left(-\frac{d^2}{dx^2} + V(x)\right) \psi(x) = E \psi(x) \quad (2)$$

were declared physical *even when* complemented by certain anomalous, complexified Dirichlet asymptotic boundary conditions [15]. Indeed, once you assume a suitable form of analyticity of $V(x)$ you may require, in a mathematically consistent manner, that

$$\psi(-\varrho \cdot e^{i\theta_{(left)}}) = \psi(+\varrho \cdot e^{i\theta_{(right)}}) = 0 \quad \varrho \rightarrow +\infty \quad (3)$$

not only for the usual $\theta_{(left)} = \theta_{(right)} = 0$ but also at *any* non-vanishing angles $\theta_{(left)} \neq 0 \neq \theta_{(right)}$.

The next and decisive step towards possible explicit and nontrivial applications of such a more or less trivial mathematics in physics has been made by Bender and Boettcher [16] who presented a persuasive numerical and semiclassical support for their conjecture that in many similar cases the spectrum *can* remain real and, hence, observable. More specifically, they employed and recommended a special choice of the conditions (3) setting, in our present notation,

$$\theta_{(left)} = -\theta_{(right)}. \quad (4)$$

They found out, purely empirically, that such a postulate seems to offer very good chances that the spectrum remains real. In the other words, in a way transferring certain numerical experience [17] and known mathematical tricks (cf., e.g., the textbooks [18] or the Buslaev’s and Grecchi’s paper [14]) into a much more ambitious

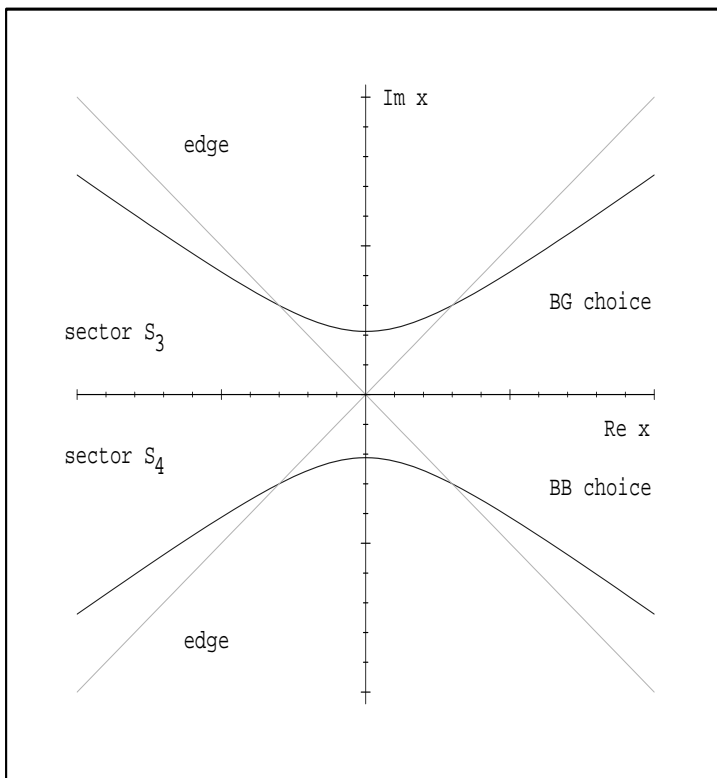


Figure 1: Complex contours of coordinates (BG = choice made in ref. [14], BB = choice made in ref. [19]).

physical project. They recommended to work with the “curved”, complex paths \mathcal{C} of coordinates x which remain left-right symmetric in the complex plane of x , i.e., which are, in an obvious sense, “ \mathcal{PT} –symmetric” (cf. their sample given here in Figure 1).

In a way fortunate for the subsequent quick development of the subject [20], the reality of spectra in many \mathcal{PT} –symmetric models has been fairly soon proved in an entirely rigorous manner [21]. The way has been opened for the birth of quantum toboggans [10].

The mathematical essence of quantum toboggans (QT, [11]) is easy to explain briefly. On an intuitive level one immediately sees that in the case of the presence of a branch point in $\psi(x)$ (say, at $x^{(BP)} = 0$), the “standard” integration paths as sampled in Figure 1 can be replaced, e.g., by the “tobogganic spiral” of Figure 2. On a more abstract level one simply has to recognize that the current (and mathematically very strong) assumptions concerning the analytic behaviour of $V(x)$ may be perceivably

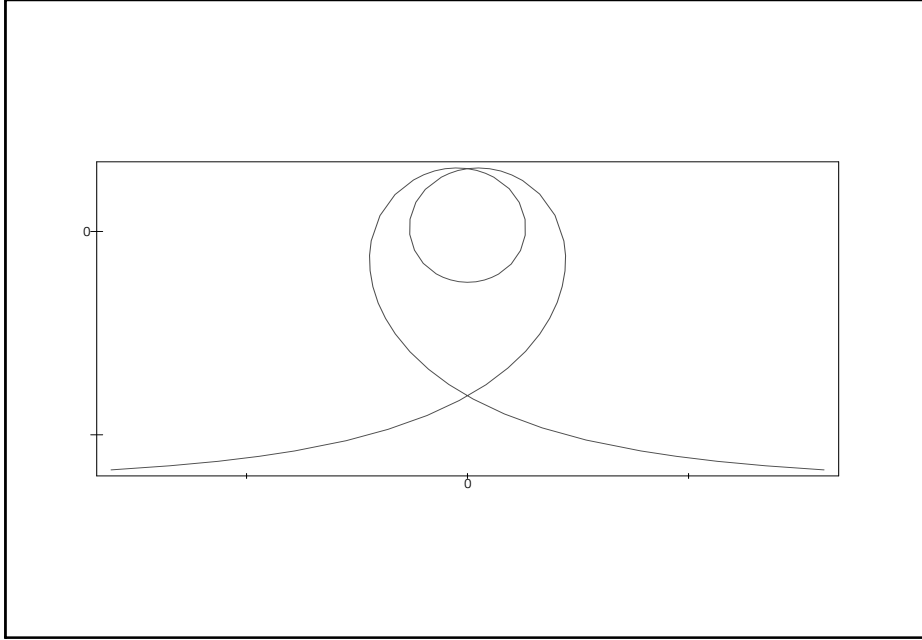


Figure 2: Tobogganic contour $\mathcal{C}^{(N)}$ with $N = 2$.

weakened without any real changes in our understanding of the \mathcal{PT} -symmetry-related problems, conjectures and constructions. *In nuce*, we may admit that in the complex plane of x our potentials may be allowed to have singularities which are distributed in a left-right symmetric manner in \mathcal{C} .

In all the cases with *more than one* branch point (cf. [11, 12] or section 4 below), the discussion of the non-equivalent QT paths would be technically complicated though still very similar to the single-branch-point simplest case. For this reason, let's now stay just in the simplest QT = QT1 case with holomorphic $V(x)$ possessing the centrifugal-like pole $\sim \ell(\ell + 1)/x^2$ in the origin. Obviously, this model generates also a branch point $x^{(BP)} = 0$ in $\psi(x)$ generated by this centrifugal-type term in the potential. In order to classify the related non-equivalent integration paths \mathcal{C} in eq. (2), one must check how many times they turn around the centre before they start approaching their $\varrho \gg 1$ asymptotics.

This check is also sufficient. Indeed, as long as the coordinates $x \in \mathcal{C}$ are complex and, by assumption, $0 = x^{(BP)} \notin \mathcal{C}$, we may just let the angles $\theta_{(left)}$ and $\theta_{(right)}$ vary *beyond* the interval $(-\pi/2, \pi/2)$. In particular, in the specific \mathcal{PT} -symmetric cases as defined by eq. (4), we shall accept a convenient convention concerning the (say, counterclockwise) orientation of the winding of the curves \mathcal{C} . Moreover, on the zeroth

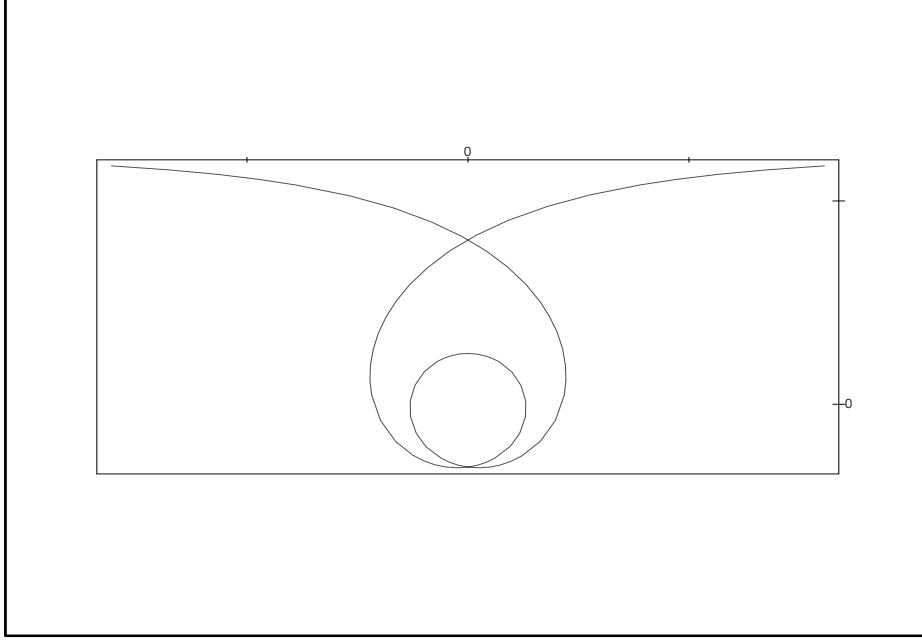


Figure 3: The complex conjugate version of the contour of Figure 2.

Riemann sheet of the total Riemann surface pertaining to our QT bound states $\psi(x)$ we shall fix our QT curves \mathcal{C} as crossing the imaginary axis below zero and in the left-right direction so that, for example, the curves running in the opposite direction (like the one displayed in Figure 3) will not be considered here as independent new models.

3 QT1 models with the single branch point

The overall winding number N of \mathcal{C} is highly relevant for the specification of the boundary conditions (3). In the presence of just single branch point $x^{(BP)}$, the knowledge of the winding number N in $\mathcal{C} = \mathcal{C}^{(N)}$ specifies the QT1 bound-state problem completely. We may visualize our QT1 spirals $\mathcal{C}^{(N)}$ as curves which are parametrized by an angle $\phi \in (-\Phi, \Phi)$ with a positive radius $\varrho = \varrho(\phi)$. In such a setting, the simplest definition of the \mathcal{PT} -symmetry of the curve $\mathcal{C}^{(N)}$ may be based on the symmetry requirement $\varrho(\phi) = \varrho(-\phi)$ [10].

For the most elementary \mathcal{PT} -symmetric harmonic-oscillator Schrödinger QT1

equation

$$\left(-\frac{d^2}{dx^2} + \frac{\ell(\ell+1)}{x^2} + x^2\right) \psi(x) = E \psi(x)$$

\mathcal{PT} -symmetric path $\mathcal{C}^{(N)}$ N -times encircles $x = 0$. At the trivial $N = 0$ this model may be defined along the straight contour

$$\mathcal{C}^{(0)} = \{x \mid x = t - i\varepsilon, t \in \mathbb{R}\}$$

and one finds that at any $\alpha(\ell) = \ell + 1/2$ it generates “twice as many” bound-state levels than its half-line predecessor (cf. [9]),

$$E = E_{n,\ell,\pm} = 4n + 2 \pm 2\alpha(\ell), \quad n = 0, 1, \dots$$

Various anharmonic non-tobogganic generalizations of this solvable model with various potentials such that $\text{Re } V(x) = +\text{Re } V(-x)$ and $\text{Im } V(x) = -\text{Im } V(-x)$ cease to be unique since even several non-tobogganic asymptotic boundary conditions

$$\psi(\pm \text{Re } L + i \text{Im } L) = 0,$$

$$|L| \gg 1 \quad \text{or} \quad |L| \rightarrow \infty.$$

can prove non-equivalent.

In the tobogganic cases characterized by a nonvanishing winding-number integer $N \neq 0$, the curves $\mathcal{C} = \mathcal{C}^{(N)}$ lie on a multisheeted Riemann surface. They can be parametrized there by an angle $\varphi \in (-(N+1)\pi, N\pi)$ as, e.g.,

$$\mathcal{C}^{(N)} = \left\{x = \varepsilon \varrho(\varphi, N) e^{i\varphi}, \varepsilon > 0\right\},$$

$$\varrho(\varphi, N) = \sqrt{1 + \tan^2 \frac{\varphi + \pi/2}{2N+1}}.$$

A deeper discussion of some features of quantum toboggans can be facilitated by the quasi-exact solvability of the underlying potential chosen, say, in the asymptotically decadic form

$$V(x) = x^{10} + \text{asymptotically smaller terms}$$

where [22]

$$\psi(x) = e^{-x^6/6 + \text{asymptotically smaller terms}}.$$

We may reparametrize

$$\psi(x) = \exp \left[-\frac{1}{6} \varrho^6 \cos 6\varphi + \dots \right]$$

and see that there exist as many as five non-tobogganic versions of this model, with angles in the eligible complex wedges

$$\begin{aligned}\Omega_{(first\ right)} &= \left(-\frac{\pi}{2} + \frac{\pi}{12}, -\frac{\pi}{2} + \frac{3\pi}{12}\right), \\ \Omega_{(first\ left)} &= \left(-\frac{\pi}{2} - \frac{\pi}{12}, -\frac{\pi}{2} - \frac{3\pi}{12}\right), \\ \Omega_{(third\ right)} &= \left(-\frac{\pi}{2} + \frac{5\pi}{12}, -\frac{\pi}{2} + \frac{7\pi}{12}\right), \quad \dots \\ \dots \quad \Omega_{(fifth\ left)} &= \left(-\frac{\pi}{2} - \frac{9\pi}{12}, -\frac{\pi}{2} - \frac{11\pi}{12}\right).\end{aligned}$$

It is worth noting that *all* of the models of this type can be interpreted as equivalent to their non-tobogganic partners confined by a *different* “effective” potential. For this purpose one can simply perform a \mathcal{PT} –symmetric change of variables in the “initial” \mathcal{PT} –symmetric model

$$\left[-\frac{d^2}{dx^2} - (ix)^2 + \lambda W(ix)\right] \psi(x) = E(\lambda) \psi(x)$$

where

$$W(ix) = \Sigma g_\beta (ix)^\beta$$

and where one sets

$$ix = (iy)^\alpha, \quad \psi(x) = y^\varrho \varphi(y).$$

Once we use the freedom in the choice of $\alpha > 0$ we have

$$i dx = i^\alpha \alpha y^{\alpha-1} dy, \quad \frac{(iy)^{1-\alpha}}{\alpha} \frac{d}{dy} = \frac{d}{dx}.$$

This gives an equivalent, “Sturmian” problem in an intermediate differential equation form

$$\begin{aligned}& y^{1-\alpha} \frac{d}{dy} y^{1-\alpha} \frac{d}{dy} y^\varrho \varphi(y) + \\ & + i^{2\alpha} \alpha^2 \left[-(iy)^{2\alpha} + \lambda W[(iy)^\alpha] - \right. \\ & \left. - E(\lambda) \right] y^\varrho \varphi(y) = 0.\end{aligned}$$

Here, the first term

$$\begin{aligned}& y^{1-\alpha} \frac{d}{dy} y^{1-\alpha} \frac{d}{dy} y^{[(\alpha-1)/2]} \varphi(y) = \\ & = y^{2+\varrho-2\alpha} \frac{d^2}{dy^2} \varphi(y) + \varrho(\varrho - \alpha) y^{\varrho-2\alpha} \varphi(y),\end{aligned}$$

“behaves” at the specific

$$\varrho = \frac{\alpha - 1}{2}$$

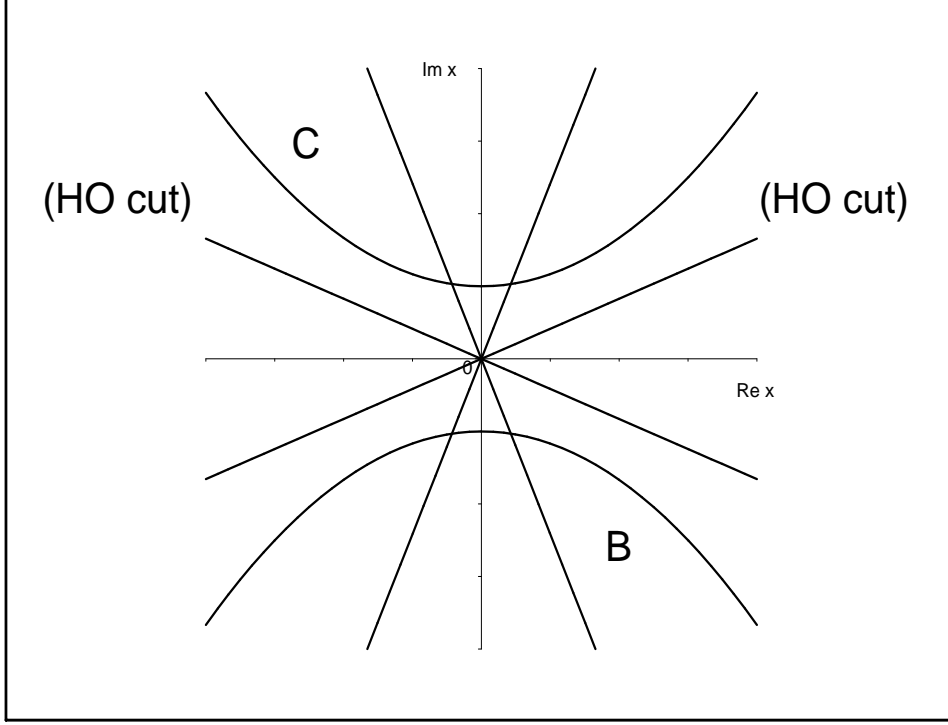


Figure 4: Sextic oscillator as a map of a \mathcal{PT} -symmetric harmonic-oscillator contour $\mathcal{C}^{(0)}$ (curve B) and of a tobogganic harmonic oscillator contour $\mathcal{C}^{(1)}$ (curve C).

so that the new equation preserves the same Schrödinger form:

$$\begin{aligned}
 & -\frac{d^2}{dy^2} \varphi(y) + \frac{\alpha^2 - 1}{4y^2} \varphi(y) + \\
 & + (iy)^{2\alpha-2} \alpha^2 \left[-(iy)^{2\alpha} + \lambda W[(iy)^\alpha] \right] \varphi(y) = \\
 & = (iy)^{2\alpha-2} \alpha^2 E(\lambda) \varphi(y).
 \end{aligned}$$

What was important is that the change of variables changed also the range of the angle in $\mathcal{C}^{(N)}$ so that by the choice of α one can diminish the winding number N . In this manner, many polynomial potentials prove interrelated. E.g., with $\alpha = 1/2$ we get the quadratic oscillator $V_g(y) = -(iy)^2 + i g_1 y + g_{-1} (iy)^{-1} + g_{-2} (iy)^{-2}$ from the sextic oscillator

$$\begin{aligned}
 & -\frac{d^2}{dx^2} \varphi(x) + \frac{\ell(\ell+1)}{x^2} \varphi(x) + V_f(x) \varphi(x) = E \varphi(x), \\
 & V_f(x) = x^6 + f_4 x^4 + f_2 x^2 + f_{-2} x^{-2},
 \end{aligned}$$

etc (cf. Figure 4).

In the conclusion of this section let us emphasize that the \mathcal{PT} -symmetry in the presence of the single branch point can be based on the introduction of the two

different parity-like operators $\mathcal{P}^{(\pm)} : x \rightarrow x \cdot \exp(\pm i\pi)$ as well as of the two eligible rotation-type innovations $\mathcal{T}^{(\pm)}$ of the time reversal.

4 QT models with more branch points

Once you study the bound-state problem

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi_n(x) + V(x) \psi_n(x) = E_n \psi_n(x), \quad (5)$$

you may set $\hbar = 2m = 1$ and choose, say, the potential with two second-order poles,

$$V(x) = V_{\text{regular}}(x) + \frac{G}{(x-1)^2} + \frac{G^*}{(x+1)^2}.$$

In this way you arrive at the wave functions

$$\psi^{(\text{general})}(x) = c_+ \psi_{(+)}^{(\text{special})}(x) + c_- \psi_{(-)}^{(\text{special})}(x) \quad (6)$$

with the two (in general, complex) branch points at, say, $x = x^{(BP)} = \pm 1$ [23]. Thus, the Riemann surface \mathcal{R} of $\psi(x)$ becomes composed of *many* sheets \mathcal{R}_k .

4.1 The harmonic-oscillator example

Without boundary conditions our differential Schrödinger equation comprises many eigenvalue problems at once [15]. In the QT = QT2 models with two branch points they are all in a one-to-one correspondence with our selection of the QT paths $x^{(\varrho)}(s)$. In a way introduced in ref. [12], they may be classified by certain “winding” or “knotting” descriptors ϱ . These $x^{(\varrho)}(s)$ connect their asymptotes while passing through a compact domain of x which contains all the singularities of $V(x)$.

In the most transparent non-tobogganic case one can stay on the single Riemann sheet and, for illustrative purposes, pick up the harmonic oscillator with $\psi_n^{(\pm|\alpha|)}(x) = x^{1/2 \pm |\alpha|} \exp(+x^2/2) \times$ a polynomial, considered as integrated along U-shaped paths

$$y^{(U)}(s) = x^{(U)}(s) + i\varepsilon = \begin{cases} |s| e^{-11i\pi/8} \exp[i\xi(s)], & s \ll -1, \\ |s| e^{3i\pi/8} \exp[i\xi(s)], & s \gg 1 \end{cases} \quad (7)$$

on which $x(s) = y(s) - i\varepsilon$, $\xi(s) \in (-\pi/8, \pi/8)$ and

$$\lim_{s \rightarrow \pm\infty} \psi[x^{(U)}(s)] = 0. \quad (8)$$

In the alternative, tobogganic cases with $N = 1$, $\xi \in (-\pi/8, \pi/8)$ and $\delta > 0$ in

$$x^{(N=1)}(s) = y^{(N=1)}(s) - i\varepsilon = \begin{cases} |s - \eta| e^{-13i\pi/8} \exp[i\xi(s)] + i\delta, & s \ll -1, \\ |s - \eta| e^{5i\pi/8} \exp[i\xi(s)] + i\delta, & s \gg 1 \end{cases} \quad (9)$$

we may consider the paths encircling two branch points by winding

- counterclockwise around $x_{(-)}^{(BP)}$ (to be marked by a letter L),
- counterclockwise around $x_{(+)}^{(BP)}$ (letter R),
- clockwise around $x_{(-)}^{(BP)}$ ($Q = L^{-1}$),
- clockwise around $x_{(+)}^{(BP)}$ ($P = R^{-1}$).

In this way, a four-letter alphabet can be used to label all paths $x = x^{(\varrho)}(s)$ by words ϱ of length $2N$ and of a concatenated form $\varrho = \Omega \cup \Omega^T$ which is due to the underlying \mathcal{PT} -symmetry $L \leftrightarrow R$.

Thus, besides the symbol $\varrho = \emptyset$ for the non-tobogganic case one has four possibilities at $N = 1$, viz.,

$$\Omega \in \{L, L^{-1}, R, R^{-1}\}, \quad N = 1,$$

$$\varrho \in \{LR, L^{-1}R^{-1}, RL, R^{-1}L^{-1}\}, \quad N = 1,$$

or the following dozen cases at $N = 2$,

$$\begin{aligned} \Omega \in \{ & LL, LR, RL, RR, L^{-1}R, R^{-1}L, LR^{-1}, \\ & RL^{-1}, L^{-1}L^{-1}, L^{-1}R^{-1}, R^{-1}L^{-1}, R^{-1}R^{-1} \} \end{aligned}$$

(with four items $LL^{-1}, L^{-1}L, RR^{-1}, R^{-1}R$ not allowed among the $4^2 = 16$ eligible ones), etc [12].

4.2 The rectification of the QT2 contours at $\varrho = \varrho_0$

In the presence of a single branch point we set

$$\mathbf{i}x = (\mathbf{i}z)^2, \quad \psi_n(x) = \sqrt{z} \varphi_n(z)$$

and remind the readers about the *strict equivalence* of the QT1 harmonic oscillator to its manifestly \mathcal{PT} -symmetric sextic-oscillator partner

$$\left(-\frac{d^2}{dz^2} + 4z^6 + 4E_n z^2 + \frac{4\alpha^2 - 1/4}{z^2} \right) \varphi(z) = 0$$

defined along a *manifestly non-tobogganic* path,

$$\mathcal{C} = \begin{cases} \sqrt{|s - \eta|} e^{-9\mathbf{i}\pi/16} \exp[\mathbf{i}\xi(s)/2] + \mathcal{O}(\delta/\eta), & s \ll -1, \\ \sqrt{|s - \eta|} e^{\mathbf{i}\pi/16} \exp[\mathbf{i}\xi(s)/2] + \mathcal{O}(\delta/\eta), & s \gg 1 \end{cases}. \quad (10)$$

In the presence of the *pair* of the branch points, say, $x^{(BP)} = \pm 1$, the simplest changes of variables can be employed again. To the Schrödinger equation

$$\left[-\frac{d^2}{dx^2} + \frac{\ell(\ell+1)}{(x-1)^2} + \frac{\ell(\ell+1)}{(x+1)^2} + V(ix) \right] \psi(x) = E \psi(x).$$

this often enables us to assign the rectified partner

$$\left[-\frac{d^2}{dz^2} + U_{eff}(iz) \right] \varphi(z) = 0$$

where

$$\begin{aligned} U_{eff}(iz) &= U(iz) + \frac{\mu(\mu+1)}{(z-1)^2} + \frac{\mu(\mu+1)}{(z+1)^2} \equiv \\ &\equiv U(iz) + 2 \frac{\mu(\mu+1)[1-(iz)^2]}{[1+(iz)^2]^2}. \end{aligned}$$

Proof

Using an implicit rectification formula

$$1 + (ix)^2 = \left[1 + (iz)^2 \right]^\kappa, \quad \kappa > 1$$

we reveal that $z = -i \varrho$ gets mapped upon itself. Hence, one can recommend the use of the explicit rectification formula $x = -i \sqrt{(1-z^2)^\kappa - 1}$. As a result, certain effective non-tobogganic potentials are obtained. Their construction is routine since

$$\frac{d}{dx} = \beta(z) \frac{d}{dz}, \quad \beta(z) = -i \frac{\sqrt{(1-z^2)^\kappa - 1}}{\kappa z (1-z^2)^{\kappa-1}}.$$

Thus, we may set $\psi(x) = \chi(z) \varphi(z)$ with $\chi(z) = \text{const} / \sqrt{\beta(z)}$ [24] and get $V_{eff}(ix) = V(iz) + 2\ell(\ell+1)[1-(ix)^2]/[1+(ix)^2]^2$ in

$$\left(-\beta(z) \frac{d}{dz} \beta(z) \frac{d}{dz} + V_{eff}[ix(z)] - E \right) \chi(z) \varphi(z) = 0.$$

or

$$U_{eff}(iz) = \frac{V_{eff}[ix(z)] - E_n}{\beta^2(z)} + \frac{\beta''(z)}{2\beta(z)} - \frac{[\beta'(z)]^2}{4\beta^2(z)}$$

in the standard Schrödinger equation. QED.

Graphically, it is interesting to reconstruct the shapes of the tobogganic pull-backs (cf. Figure 5 for illustration). For this purpose, in the vicinity of the negative imaginary axis we may consider the mapping

$$z = -i r e^{i\theta} \longrightarrow x = -i \left[(1 + r^2 e^{2i\theta})^\kappa - 1 \right]^{1/2}.$$

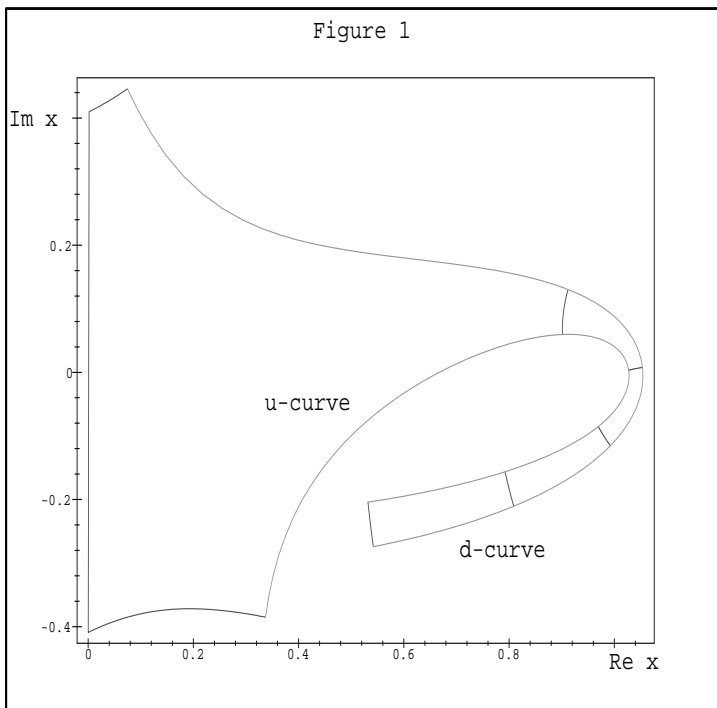


Figure 5: Two bitoboggans ($\kappa = 2.4$, $s \in (0.4, 1.4)$).

At the small radii r it degenerates to the mere multiplication by a constant $\sqrt{\kappa}$. At the larger radii we arrive at a more complicated knot-like shapes of $x^{\epsilon_0}(s)$, tractable easily by computer graphics, via their definition as a pullback of the straight-line $z(s) = s - i\varepsilon$. In this approach the winding number N proves fairly sensitive to the value of shift ε . In Figure 5 we see the clear acceleration of the winding after transition from $\varepsilon = \varepsilon_u = 0.15$ to $\varepsilon = \varepsilon_d = 0.20$.

5 Summary

In the context of mathematics and in a way paralleling the birth of interest in \mathcal{PT} symmetry, the concept of quantum toboggans could in fact be also understood as almost trivial in mathematics. In physics, the first hints for its introduction resulted from several independent mathematical sources. The first one has been the above-cited Buslaev's and Grecchi's paper [14] where a *formal necessity* of a branch point involved *all* their analytic bound-state wave functions $\psi_n(x)$. A clearer understanding of this hint (reflecting the necessity of an effective kinematical centrifugal

force in more dimensions) came a few years later when attention has been turned to an “unperturbed” harmonic-oscillator special case of their model [9]. In parallel, virtually the same branch points in wave functions resurfaced during the proofs of the reality of the energies [21], during the studies of the quasi-exact solvability of certain \mathcal{PT} –symmetric models [22] and after a supersymmetrization of certain \mathcal{PT} –symmetric Hamiltonians [23].

Let us summarize: what should be remembered in the context of mathematics is the new use of the changes of variables in Schrödinger equations. This can rectify the QT paths of coordinates and may also lead to some new and nonstandard *feasible* calculations.

In the parallel phenomenological model-building context, the perspective of new physics may be expected to be derived from the prospective use of the tobogganic paths $\mathcal{C}^{(N)}$. This could throw new light not only on “innovated” bound states (of a “topological” origin) but also on the very unusual scattering-type states [12].

Figure captions

Figure 1. Complex contours of coordinates (BG = choice made in ref. [14], BB = choice made in ref. [19]).

Figure 2. Tobogganic contour $\mathcal{C}^{(N)}$ with $N = 2$.

Figure 3. The complex conjugate version of the contour of Figure 2.

Figure 4. Sextic oscillator as a map of a \mathcal{PT} –symmetric harmonic-oscillator contour $\mathcal{C}^{(0)}$ (curve B) and of a tobogganic harmonic oscillator contour $\mathcal{C}^{(1)}$ (curve C).

Figure 5. Two bitoboggans ($\kappa = 2.4$, $s \in (0.4, 1.4)$).

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